

ON THE CARDINALITY OF ALMOST DISCRETELY LINDELÖF SPACES

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ABSTRACT. We prove that every almost discretely Lindelöf first-countable Hausdorff space has cardinality at most continuum in ZFC, thus completely answering Question 4.5 from [11] and a question implicitly asked by Juhász, Soukup and Szentmiklóssy in [10]. Using a different argument we prove that under $2^{<\mathfrak{c}} = \mathfrak{c}$ (which is a consequence of Martin's Axiom, for example), every sequential almost discretely Lindelöf Hausdorff space of pseudcharacter at most continuum has cardinality at most continuum. We conclude with a few related results and questions.

1. INTRODUCTION

In [1] Arhangel'skii proved his celebrated theorem stating that every Lindelöf first-countable Hausdorff space has cardinality at most continuum. Besides solving a long standing question due to Alexandroff and Urysohn, Arhangel'skii's Theorem gave a definite boost to the area of cardinal invariants in topology, inspiring new techniques, results and questions that continue to be the object of current research (see [8] for a survey on Arhangel'skii's Theorem and its legacy).

Recall that space is said to be *discretely Lindelöf* if the closure of every discrete set is Lindelöf. A well-known question due to Arhangel'skii asks whether every regular discretely Lindelöf space is Lindelöf.

A space is defined to be *almost discretely Lindelöf* [11] if for every discrete set $D \subset X$ there is a Lindelöf subspace L of X such that $D \subset L$. Of course every discretely Lindelöf space is almost discretely Lindelöf. Any example of an S -space (a regular hereditarily separable non-Lindelöf space) provides an (alas, only consistent) example of an almost discretely Lindelöf non-Lindelöf regular space. It is still open whether there exists an example of such a space in ZFC.

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The authors of [11] prove that every almost discretely Lindelöf first-countable space has cardinality at most $2^{\mathfrak{c}}$ and ask whether every almost discretely Lindelöf first-countable Hausdorff space has cardinality bounded by the continuum. We prove that this is the case in ZFC (Juhász, Soukup and Szentmiklóssy [10] proved that for regular spaces and asked whether it is also true for Hausdorff spaces).

By a completely different argument, we prove that if $2^{<\mathfrak{c}} = \mathfrak{c}$ then: 1) every almost discretely Lindelöf sequential space such that $\psi(X) \leq \mathfrak{c}$ has cardinality at most continuum and 2) every almost discretely Lindelöf Hausdorff space such that $\psi_c(X) \cdot t(X) = \omega$ has cardinality at most continuum. It remains open whether these results can be proved in ZFC or whether in 2) the closed pseudocharacter can be replaced with the pseudocharacter, even consistently.

We conclude by exploring a few further generalizations and related results.

In our proofs we will often use elementary submodels of the structure $(H(\mu), \epsilon)$. Readers who are not familiar with elementary submodels can find a rigorous treatment of them in Kunen's book [12] and an informal introduction to their applications to topology in Dow's survey [4].

All spaces under consideration are assumed to be T_1 . Undefined notions can be found in [7] for topology and [12] for set theory. Our notation regarding cardinal functions mostly follows [9]. In particular, $\psi(X)$ and $t(X)$, denote the pseudocharacter and tightness of X respectively. We recall the definition of these two important cardinal functions, given that they are essential for many of the results in our paper.

Let A be a subset of X . The *pseudocharacter of A in X* ($\psi(A, X)$) is defined as the minimum cardinal κ such that A is the intersection of κ many open sets. We denote $\psi(\{x\}, X)$ by $\psi(x, X)$. The pseudocharacter of the space X is defined as $\psi(X) = \sup\{\psi(x, X) : x \in X\}$.

The *tightness of the point x in the space X* ($t(x, X)$) is defined as the minimum cardinal κ such that for every set $A \subset X$ with $x \in \overline{A} \setminus A$ there is a κ -sized set $B \subset A$ such that $x \in \overline{B}$. The tightness of the space X is defined as $t(X) = \sup\{t(x, X) : x \in X\}$.

2. THE MAIN RESULT

Recall that a space is right-separated if and only if it admits a well-ordering where every initial segment is open. It is well-known and easy to prove that a space is right-separated if and only if it is *scattered* (that is, every non-empty subset contains an isolated point). We denote by $h(X)$ the supremum of the cardinalities of the *right-separated* subsets

of X . As is well known (see, for example, [9], 2.9), $hL(X) = h(X)$ for every space X , where $hL(X)$ denotes the hereditarily Lindelöf degree of X .

We denote by $g(X)$ the supremum of the cardinalities of the closures of discrete sets in X . Since every scattered space has a dense discrete subset we have $h(X) \leq g(X)$.

We start with an observation contained in the proof of Theorem 4 from [10] which we would like to isolate for the convenience of the reader.

Lemma 1. [10] *Let X be an almost discretely Lindelöf T_2 space X . Then $hL(X) \leq 2^{x(X)}$.*

Proof. Recall (see for example 2.5 of [9]) that $|Y| \leq d(Y)^{x(Y)}$ for every Hausdorff space Y and hence $|\overline{D}| \leq |D|^{x(X)}$. But since X is almost discretely Lindelöf, for every discrete set $D \subset X$, there is a Lindelöf space $L \subset X$ such that $D \subset L$. It follows that $|\overline{D}| \leq 2^{x(X)}$, by Arhangel'skii's Theorem. Taking suprema we obtain that $hL(X) = h(X) \leq 2^{x(X)}$. \square

Definition 2. [15] *For any space X and any set $A \subseteq X$, $Cl_\theta(A)$ is the set of all points x such that $\overline{U} \cap A \neq \emptyset$ for every neighbourhood U of x . A is θ -closed if $A = Cl_\theta(A)$. The θ -closure $[A]_\theta$ is the smallest θ -closed set containing A .*

Lemma 3. *Let X be an almost discretely Lindelöf Hausdorff space, and A be a θ -closed set. Then $\psi(A, X) \leq 2^{x(X)}$, that is, A is the intersection of a family of $2^{x(X)}$ open sets.*

Proof. For each $x \in X \setminus A$ we may fix an open neighbourhood U_x of x such that $\overline{U_x} \cap A = \emptyset$. By Lemma 1, there is a set $S \subseteq X \setminus A$ such that $|S| \leq 2^{x(X)}$ and $\bigcup \{U_x : x \in S\} = X \setminus A$. Then $A = \bigcap \{X \setminus \overline{U_x} : x \in S\}$. \square

Definition 4. *We say that a sequence $\{x_\alpha : \alpha < \kappa\}$ is θ -free if:*

$$[\{x_\alpha : \alpha < \beta\}]_\theta \cap \overline{\{x_\alpha : \beta \leq \alpha < \kappa\}} = \emptyset$$

for every $\beta < \kappa$. The cardinal function $F_\theta(X)$ denotes the supremum of the cardinalities of all θ -free sequences contained in X .

Recall that a sequence $\{x_\alpha : \alpha < \kappa\}$ is called *free* if

$$\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \leq \alpha < \kappa\}} = \emptyset$$

for every $\beta < \kappa$. The cardinal function $F(X)$ is defined as the supremum of the cardinalities of free sequences contained in X . Free sequences are an important tool in Arhangel'skii's solution of the Alexandroff-Urysohn problem.

Obviously, every θ -free sequence is free, so $F_\theta(X) \leq F(X)$ and $F(X) = F_\theta(X)$ for every regular space X . However, the inequality may be strict for non-regular spaces (see Section 4).

The following lemma is proved via a simple standard argument.

Lemma 5. *In an almost discretely Lindelöf space X , every free sequence has length at most $\chi(X)$.*

Recall that the *closed pseudocharacter of the point x in the space X* ($\psi_c(x, X)$) is defined as the minimum cardinal κ such that there is a family $\{U_\alpha : \alpha < \kappa\}$ of neighbourhoods of x such that $\bigcap \{\overline{U_\alpha} : \alpha < \kappa\} = \{x\}$. The closed pseudocharacter of the space X is then defined as $\psi_c(X) = \sup\{\psi_c(x, X) : x \in X\}$. It is easy to see that $\psi_c(X) \leq \chi(X)$, for every Hausdorff space X . The proof of the following theorem is a variation on the proof of Theorem 2.3 from [2].

Theorem 6. *Let X be a space such that $F_\theta(X) \cdot \psi_c(X) \leq \kappa$ and $\psi([F]_\theta, X) \leq 2^\kappa$, for every θ -free sequence $F \subset X$. Then $|X| \leq 2^\kappa$.*

Proof. Let μ be a regular cardinal large enough so that $H(\mu)$ contains all we need and reflects the finite number of formulas needed for the proof. Let M be a κ -closed elementary submodel of $H(\mu)$ such that $X \in M$, $|M| = 2^\kappa$ and $2^\kappa + 1 \subset M$.

Claim 1. Let F be a θ -free sequence in X contained in $X \cap M$ and let p be a point outside of $[F]_\theta$. Then there is an open set $U \in M$ such that $[F]_\theta \subset U$ and $p \notin U$.

Proof of Claim 1. Since F is a θ -free sequence in X , the cardinality of F does not exceed κ , so $F \in M$. Hence, by elementarity, we also have that $[F]_\theta \in M$. Now $\psi([F]_\theta, X) \leq 2^\kappa$, so we can fix an open family $\mathcal{U} \in M$ having cardinality 2^κ , such that $[F]_\theta = \bigcap \mathcal{U}$. Note that $\mathcal{U} \subset M$. Now pick $U \in \mathcal{U}$ such that $p \notin U$ and note that U satisfies the requirement of Claim 1. \triangle

Claim 2. Let Z be any subset of $X \cap M$ and x be a point outside of the θ -closure of Z . Then there is a κ -sized open family \mathcal{V} , contained in M , such that $Z \subset \bigcup \mathcal{V} \subset X \setminus \{x\}$.

Proof of Claim 2. Suppose that the statement of the claim is false. Then we can inductively find points $\{x_\alpha : \alpha < \kappa^+\} \subset Z$ and open sets $\{U_\alpha : \alpha < \kappa^+\} \subset M$ such that:

- (1) $[\{x_\alpha : \alpha < \beta\}]_\theta \subset U_\beta$.
- (2) $x \notin U_\beta$.
- (3) $x_\beta \notin \bigcup\{U_\alpha : \alpha \leq \beta\}$, for every $\beta < \kappa^+$.

To see that, suppose for a given $\delta < \kappa^+$ we have constructed $\{(x_\alpha, U_\alpha) : \alpha < \delta\}$ satisfying the three conditions above up to δ . Note that $\{x_\alpha : \alpha < \delta\}$ is a θ -free sequence. Indeed, from the first and third condition it follows that $[\{x_\alpha : \alpha < \gamma\}]_\theta \subset U_\gamma$ and $\{x_\alpha : \gamma \leq \alpha < \delta\} \subset X \setminus U_\gamma$, which implies that $[\{x_\alpha : \alpha < \gamma\}]_\theta \cap \overline{\{x_\alpha : \gamma \leq \alpha < \delta\}} = \emptyset$.

Therefore, we can use Claim 1 to find an open set U_δ such that $[\{x_\alpha : \alpha < \delta\}]_\theta \subset U_\delta$ and $x \notin U_\delta$.

From the assumption that the statement of Claim 2 is false, we are now allowed to pick a point $x_\delta \in Z \setminus \bigcup\{U_\alpha : \alpha \leq \delta\}$ and thus continue the induction.

It is easy to see that eventually $\{x_\alpha : \alpha < \kappa^+\}$ is a θ -free sequence in X of cardinality κ^+ , which contradicts $F_\theta(X) \leq \kappa$. \triangle

Claim 3. $X \subset M$.

Proof of Claim 3. Suppose the contrary and let z be a point outside of M . Let $\{V_\alpha : \alpha < \kappa\}$ be open neighbourhoods of z such that $\bigcap\{\overline{V_\alpha} : \alpha < \kappa\} = \{z\}$. Apply Claim 2 to $Z_\alpha = (X \cap M) \setminus \overline{V_\alpha}$ to obtain a κ -sized open family $\mathcal{W}_\alpha \subset M$ such that $Z_\alpha \subset \bigcup \mathcal{W}_\alpha \subset X \setminus \{z\}$. Let $\mathcal{W} = \bigcup\{\mathcal{W}_\alpha : \alpha < \kappa\}$. Note that $\mathcal{W} \in M$ by κ -closedness of M and $X \cap M \subset \bigcup \mathcal{W}$, which implies $M \models X \subset \bigcup \mathcal{W}$, and hence $H(\mu) \models X \subset \bigcup \mathcal{W}$. But that contradicts the fact that $z \notin \bigcup \mathcal{W}$. \triangle

□

The following corollary completely answers Question 4.5 from [11] and a question implicitly asked in [10].

Corollary 7. *Let X be an almost discretely Lindelöf Hausdorff space. Then $|X| \leq 2^{\chi(X)}$.*

Juhász, Soukup and Szentmiklóssy proved in [10] that every almost discretely Lindelöf space has cardinality at most $2^{\chi(X)}$ with the additional assumption that X is a regular space.

3. FROM THE CHARACTER TO THE PSEUDOCHARACTER

One of the most interesting early refinements of Arhangel'skii's Theorem is the one due to Arhangel'skii and Shapirovskii (see [13]) in which the character is replaced by the product of the tightness and the pseudocharacter. This result motivated Arhangel'skii's problem asking whether there exists a Lindelöf space with countable pseudocharacter

having cardinality larger than the continuum. Although consistent counterexamples were constructed by Dow [5], Gorelic [6] and Shelah [14], this question is still open in ZFC.

In view of the Arhangel'skii-Shapirovsii result, it is natural to ask whether the character can be replaced with the product of the pseudocharacter and the tightness in our main result and we will now offer a few partial answers.

Recall that a space is sequential if every non-closed set contains a sequence converging outside of it. It is easy to see that a closure of a subspace in a sequential space is obtained by iterating the sequential closure at most ω_1 many times. Every first-countable space is sequential and every sequential space has countable tightness.

Theorem 8. ($2^{<\mathfrak{c}} = \mathfrak{c}$). *Let X be a T_2 sequential almost discretely Lindelöf space such that $\psi(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$.*

Proof. Let μ be a large enough regular cardinal. Let M be a $< \mathfrak{c}$ -closed elementary submodel of $H(\mu)$ such that $|M| = \mathfrak{c}$, $\mathfrak{c} + 1 \subset M$ and $X \in M$.

From the fact that X is a Hausdorff sequential space and the fact that M is ω -closed it follows that $X \cap M$ is a closed subset of X .

We claim that $d(X) \leq \mathfrak{c}$, and that would finish the proof, because every sequential space of density continuum has cardinality continuum.

Suppose by contradiction that $d(X) \geq \mathfrak{c}^+$. Using that, it is easy to find a left-separated subset L of X having cardinality \mathfrak{c}^+ . Without loss we can assume that $L \in M$.

Since L has cardinality larger than the continuum, we can pick a point $p \in L \setminus M$. Fix a point $x \in X \cap M$. Then can find a family $\mathcal{U}_x \in M$ of cardinality continuum such that $\bigcap \mathcal{U}_x = \{x\}$. Since $|\mathcal{U}_x| \leq \mathfrak{c}$ and $\mathfrak{c} + 1 \subset M$ we actually have that $\mathcal{U}_x \subset M$. Hence, for every $x \in X \cap M$, we can find an open set $U_x \in M$ such that $x \in U_x$ and $p \notin U_x$.

Now $\mathcal{U} = \{U_x : x \in X \cap M\}$ is an open cover of $X \cap M$.

Claim. There is a $< \mathfrak{c}$ -sized subcollection of \mathcal{U} covering $L \cap M$.

Proof of Claim. If $L \cap M$ had cardinality smaller than the continuum, this would be trivially true. So we can assume that $|L \cap M| = \mathfrak{c}$.

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of \mathcal{U} in type \mathfrak{c} and set $V_\alpha = U_\alpha \setminus \bigcup \{U_\beta : \beta < \alpha\}$. Suppose by contradiction that the statement of the Claim is not true. Then the set $S = \{\alpha < \mathfrak{c} : V_\alpha \cap (L \cap M) \neq \emptyset\}$ has cardinality continuum.

Pick a point $x_\alpha \in V_\alpha \cap L$, for every $\alpha \in S$. Then $R = \{x_\alpha : \alpha \in S\}$ is a set of size continuum which is both right-separated and left-separated.

So by 2.12 of [9] the set R contains a discrete set D having cardinality continuum.

Since X is almost discretely Lindelöf, we can find a Lindelöf subspace $Y \subset X$ such that $D \subset Y$. Now, $X \cap M$ being closed, the set $Y \cap M$ is also Lindelöf, and since \mathcal{U} covers $Y \cap M$, we can find an ordinal $\delta < \mathfrak{c}$ such that $D \subset Y \cap M \subset \bigcup \{U_\alpha : \alpha < \delta\}$. But since D has cardinality continuum, there must be $\gamma > \delta$ such that $D \cap V_\gamma \neq \emptyset$ and this contradicts the fact that V_γ is disjoint from $\bigcup \{U_\alpha : \alpha < \delta\}$. \triangle

Fix a subcollection $\mathcal{V} \subset \mathcal{U}$ of cardinality smaller than the continuum such that $L \cap M \subset \bigcup \mathcal{V}$.

Since M is $< \mathfrak{c}$ -closed we have that $\mathcal{V} \in M$ and since L is also an element of M it follows that: $M \models L \subset \bigcup \mathcal{V}$.

By elementarity $H(\mu) \models L \subset \bigcup \mathcal{V}$, but that is a contradiction because $p \in L \setminus \bigcup \mathcal{V}$. \square

Theorem 9. ($2^{<\mathfrak{c}} = \mathfrak{c}$) *Let X be an almost discretely Lindelöf Hausdorff space such that $\psi_c(X) \cdot t(X) = \omega$. Then $|X| \leq \mathfrak{c}$.*

Proof. Let M be given as in the proof of Theorem 8. The proof is essentially the same as the proof of Theorem 8, except that the argument proving that $X \cap M$ is closed is different. Here it is. Let $x \in \overline{X \cap M}$. Pick a family $\{U_n : n < \omega\}$ of neighbourhoods of x such that $\bigcap \{\overline{U_n} : n < \omega\} = \{x\}$. Use the fact that the tightness of X is countable to pick a countable set $C \subset X \cap M$ such that $x \in \overline{C}$. Note that, since M is ω -closed the set $U_n \cap C$ belongs to M , for every $n < \omega$. From elementarity and ω -closedness of M again it follows that $\{x\} = \bigcap \{\overline{U_n \cap C} : n < \omega\}$ is also an element of M . Hence $x \in X \cap M$ and this concludes the proof that $X \cap M$ is closed. \square

Question 1. *Are Theorems 8 and 9 true in ZFC?*

4. ODDS AND ENDS

A *cellular family* is a family of pairwise disjoint non-empty open sets. The cellularity of X ($c(X)$) is defined as the supremum of the cardinalities of the cellular families in X .

The following is a natural generalization of the notion of an almost discretely Lindelöf space.

Definition 10. *We define a space X to be cellular-Lindelöf if for every cellular family \mathcal{U} there is a Lindelöf subspace $L \subset X$ such that $U \cap L \neq \emptyset$, for every $U \in \mathcal{U}$.*

The next proposition follows immediately from the definition.

Proposition 11.

- (1) *Every ccc space is cellular-Lindelöf.*
- (2) *Every Lindelöf space is cellular-Lindelöf.*
- (3) *Every almost discretely Lindelöf space is cellular-Lindelöf.*

So the cellular-Lindelöf property turns out to be a common weakening of the countable chain condition and the Lindelöf property, in a similar vein as the weak Lindelöf property (see [3]).

Theorem 12. *Let X be a Hausdorff cellular-Lindelöf first-countable space. Then $c(X) \leq \mathfrak{c}$.*

Proof. Let \mathcal{U} be a cellular family. Since X is cellular-Lindelöf we can find a Lindelöf space $L \subset X$ such that $L \cap U \neq \emptyset$, for every $U \in \mathcal{U}$. But every Lindelöf first-countable space has cardinality at most the continuum, so $|L| \leq \mathfrak{c}$ and hence $|\mathcal{U}| \leq \mathfrak{c}$. \square

Corollary 13. *Every Hausdorff cellular-Lindelöf first countable space has cardinality at most $2^{\mathfrak{c}}$.*

Proof. This is an immediate consequence of the Hajnal-Juhász inequality $|X| \leq 2^{X(X) \cdot c(X)}$ (see, for example, [9], 2.15 b)) and Theorem 12. \square

Example 14. *There are cellular-Lindelöf non-linearly Lindelöf Tychonoff spaces in ZFC.*

Proof. Let $X = \Sigma(2^{\aleph_0}) = \{x \in 2^{\aleph_0} : |x^{-1}(1)| \leq \aleph_0\}$ with the topology induced from the usual product topology on 2^{\aleph_0} . Then X is ccc and hence it's cellular Lindelöf. Moreover X is countably compact, so it can't be linearly Lindelöf, or otherwise it would be compact. \square

The above example should be contrasted with the fact that no example of a regular almost discretely Lindelöf non-Lindelöf space is known in ZFC. Also it seems that not even a consistent T_1 example of a discretely Lindelöf non-Lindelöf space is known at the moment.

Recall that a space is weakly Lindelöf if every open cover has a countable subcollection with a dense union (see [3]).

Example 15. *There is a weakly Lindelöf T_2 non-cellular Lindelöf space.*

Proof. This is Example 2.3 from [3]. Let κ be a cardinal larger than the continuum and let A be a countable dense subset of the irrationals. Define a topology on $X = (\mathbb{Q} \times \kappa) \cup A$ by declaring a basic neighbourhood of a point (x, α) , where $x \in \mathbb{Q}$ and $\alpha < \kappa$, to be $(U \cap \mathbb{Q}) \times \{\alpha\}$, where U is an open Euclidean interval containing x and a basic neighbourhood of a point $y \in A$ to be of the form $(U \cap A) \cup ((U \cap \mathbb{Q}) \times \kappa)$.

This space is weakly Lindelöf because every open set containing A is dense in X (see [3]). However, it is not cellular-Lindelöf because it is first-countable but $c(X) > \mathfrak{c}$. \square

Note that Example 15 is only Hausdorff.

Question 2. *Is there a Tychonoff example of a weakly Lindelöf non-cellular Lindelöf space?*

Question 3. *Is there a cellular-Lindelöf non-weakly Lindelöf space?*

Question 4. *Is it true that every first-countable cellular-Lindelöf Hausdorff space has cardinality at most continuum?*

We conclude with a few more applications of the notion of θ -free sequence. But first of all, let us note how the inequality $F_\theta(X) < F(X)$ can occur even for *Urysohn spaces* (that is spaces where each pair of distinct points can be separated by open neighbourhoods with disjoint closures).

Example 16. *A Urysohn space X such that $F_\theta(X) < F(X)$.*

Proof. Let $X = K(\omega)$ be the Katětov extension of ω . Recall that the underlying set of $K(\omega)$ is the same of the Čech-Stone compactification of the integers $\beta\omega$, that is, the set of all ultrafilters on ω (principal ultrafilters are identified with points of ω in the obvious way) but a local base at $p \in K(\omega)$ in the Katětov extension is given by $\{\{p\} \cup A : A \in p\}$. Note that $K(\omega) \setminus \omega$ is a closed discrete set of cardinality $2^{\mathfrak{c}}$. Therefore, $F(K(\omega)) = 2^{\mathfrak{c}}$.

The topology of $K(\omega)$ is finer than the topology of $\beta\omega$, however, for every $p \in K(\omega) \setminus \omega$, the topologies induced on $\{p\} \cup \omega$ by $K(\omega)$ and $\beta\omega$ are the same. Combining this with the observation that ω is dense in $K(\omega)$ we see that, for every open set $U \subset K(\omega)$, we have $\overline{U} = Cl_{K(\omega)}(U) = Cl_{K(\omega)}(U \cap \omega) = Cl_{\beta\omega}(U \cap \omega)$. Therefore the closure of an open set in $K(\omega)$ is actually a clopen set in $\beta\omega$.

Let $S = \{x_\alpha : \alpha < \kappa\}$ be a θ -free sequence in $K(\omega)$. Fix $\alpha < \kappa$. For every $\gamma \in \kappa \setminus \alpha$ there exists an open neighbourhood U_γ of x_γ such that $\overline{U_\gamma} \cap \{x_\beta : \beta < \alpha\} = \emptyset$. Since the set $\bigcup \{\overline{U_\gamma} : \gamma \in \kappa \setminus \alpha\}$ is (cl)open in $\beta\omega$, we see that S is a left-separated set in $\beta\omega$. Thus, we have that $|S| \leq \mathfrak{c}$ and hence $F_\theta(K(\omega)) \leq \mathfrak{c}$. \square

Given a space X , a set $A \subseteq X$ is θ -dense in X if $[A]_\theta = X$. The θ -density $\theta d(X)$ is the smallest cardinality of a θ -dense subset of X .

Theorem 17. *Let X be a space. If a cardinal λ satisfying $F_\theta(X) < \lambda \leq (2^{F_\theta(X)})^+$ is a caliber of X , then $\theta d(X) \leq 2^{F_\theta(X)}$.*

Proof. Let $F_\theta(X) = \kappa$ and assume by contradiction that the θ -density of X is bigger than 2^κ . Fix a choice function $\eta : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$. We will define by induction an increasing family $\{A_\alpha : \alpha < \lambda\}$ of subsets of X of cardinality not exceeding 2^κ and a family $\{U_\alpha : \alpha < \lambda\}$ of non-empty open subsets of X in such a way that:

- (1) $[A_\alpha]_\theta \cap \overline{U_\alpha} = \emptyset$;
- (2) if $\mathcal{V} \subseteq \{U_\beta : \beta < \alpha\}$ satisfies $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \neq \emptyset$, then $\eta(\bigcap \mathcal{V}) \in A_\alpha$.

To justify the inductive construction, let us assume to have already defined the sets $\{A_\beta : \beta < \alpha\}$ and $\{U_\beta : \beta < \alpha\}$. Since $\alpha < \lambda$ and $\lambda \leq (2^{F_\theta(X)})^+$, we have $|\{U_\beta : \beta < \alpha\}| \leq |\alpha| \leq 2^\kappa$. Consequently, the set $B = \{\eta(\bigcap \mathcal{V}) : \mathcal{V} \subseteq \{U_\beta : \beta < \alpha\}, |\mathcal{V}| \leq \kappa \text{ and } \bigcap \mathcal{V} \neq \emptyset\}$ has cardinality not exceeding 2^κ . Then, let $A_\alpha = B \cup \bigcup \{A_\beta : \beta < \alpha\}$. As we are assuming that the θ -density of X is bigger than 2^κ , we may find a non-empty open set U_α such that $[A_\alpha]_\theta \cap \overline{U_\alpha} = \emptyset$.

Since λ is a caliber of X , there exists a set $S \subseteq \lambda$ such that $|S| = \lambda$ and $\bigcap \{U_\alpha : \alpha \in S\} \neq \emptyset$. We may fix an increasing mapping $f : \lambda \rightarrow S$. Observe now that we are assuming $\kappa^+ \leq \lambda$. For any $\alpha < \kappa^+$ let $x_\alpha = \eta(\bigcap \{U_{f(\xi)} : \xi \leq \alpha\})$. We claim that the set $\{x_\alpha : \alpha < \kappa^+\}$ so obtained is a θ -free sequence in X . To check this, fix $\alpha < \kappa^+$ and observe that for each $\beta < \alpha$ we have $x_\beta \in A_{f(\beta)+1} \subseteq A_{f(\alpha)}$. Moreover, for each $\beta \geq \alpha$ the set $U_{f(\alpha)}$ occurs in the definition of x_β and consequently $x_\beta \in U_{f(\alpha)}$. This means that $\{x_\beta : \beta < \alpha\} \subseteq A_{f(\alpha)}$ and $\{x_\beta : \alpha \leq \beta < \kappa^+\} \subseteq U_{f(\alpha)}$. Therefore $[\{x_\beta : \beta < \alpha\}]_\theta \cap \overline{\{x_\beta : \alpha \leq \beta < \kappa^+\}} \subseteq [A_{f(\alpha)}]_\theta \cap \overline{U_{f(\alpha)}} = \emptyset$. The validity of the claim contradicts the hypothesis and the proof is then complete. \square

Corollary 18. *Let X be a regular space. If a cardinal λ satisfying $F(X) < \lambda \leq (2^{F(X)})^+$ is a caliber of X , then $d(X) \leq 2^{F(X)}$*

Corollary 19. *Let X be a regular sequential space with no uncountable free sequence and $\lambda \leq \mathfrak{c}^+$ be an uncountable cardinal such that λ is a caliber of X . Then $|X| \leq \mathfrak{c}$.*

Finally the argument proving Theorem 6 can be used to give a similar estimate to the θ -density of a space, but replacing the caliber assumption with an assumption about the pseudocharacter of θ -closures of free sequences.

Proposition 20. *Let X be a space such that $F_\theta(X) \leq \kappa$ and $\psi([F]_\theta, X) \leq 2^\kappa$, for every θ -free sequence F . Then $\theta d(X) \leq 2^\kappa$.*

Proof. Let M be as in the proof of Theorem 6. Note that we never used the assumption $\psi(X) \leq \kappa$ in the proof of Claims 1 and 2. We will show that $X \cap M$ is θ -dense in X . Indeed, suppose that this is not the case, and let y be a point outside of $[X \cap M]_\theta$. Using Claim 2 from the proof of Theorem 6 with $Z = X \cap M$, we can find a κ -sized family $\mathcal{W} \subset M$ such that $[X \cap M]_\theta \subset \bigcup \mathcal{W}$ and $y \notin \bigcup \mathcal{W}$. Since M is κ -closed we have $\mathcal{W} \in M$, therefore $M \models X \subset \bigcup \mathcal{W}$, whence, by elementarity, $H(\theta) \models X \subset \bigcup \mathcal{W}$, but that contradicts the fact that $y \notin \bigcup \mathcal{W}$. \square

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